Affine and cyclotomic BMW algebras

Iowa City, March 2011

Frederick Goodman

University of Iowa goodman@math.uiowa.edu

Introduction

This work concerns affine and cyclotomic BMW algebras. These are BMW analogues of affine and cyclotomic Hecke algebras.

Acknowledgements/ related work

Part of the work presented here is joint with **Holly Hauschild Mosley**.

Two other groups have studied the same subject and there is considerable overlap between their work and ours: (1) **Stewart Wilcox and Shona Yu** and (2) **Hebing Rui, Mei Si, and Jie Xu**.

There are closely related algebras called <u>degenerate affine and cyclotomic BMW algebras</u> introduced by **Nazarov** and studied by **Ariki, Mathas and Rui**.

Arun Ram and Rosa Orellana have studied representations of affine BMW algebras via a Lie theory construction; and **Arun, Zajj Daugherty, and Rahbar "rv" Virk** have work in progress on the subject, about which Zajj is speaking on Sunday.

Review of ordinary BMW algebras

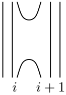
- Recall that the Kauffman link invariant is determined by skein relations (on framed links in S^3)
  1. (Crossing relation)  −  $=(q^{-1}-q)\Big($  −  $\Big)$.
  2. (Untwisting relation)  $=\rho \;\Big|$ and  $=\rho^{-1} \;\Big|$.
  3. (Free loop relation) $L\cup\bigcirc=\delta L$, where $L\cup\bigcirc$ is the union of a link L and an additional closed loop with zero framing.
- Here ρ, q and δ are elements of some integral domain R such that
$$\rho^{-1}-\rho=(q^{-1}-q)(\delta-1).$$
- The existence of the Kauffman invariant is equivalent to the skein module of links modulo the Kauffman relations being free over R of rank 1.

Review of ordinary BMW algebras, cont.

- Birman & Wenzl, and independently Murakami, invented a quotient of the braid group algebra from which the Kauffman invariant could be derived in the same manner as the 2–variable Jones invariant (HOMFLYPT) comes from the Hecke algebra.
- The definition (by generators and relations) is on the next page. If you are not already familiar with it, you can't possibly take it in.
- The only thing you need to notice is that there are generators g_i

  which should be thought of as braid generators 

  and

  generators e_i which should be thought of as tangles 

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  and there are a lot of relations which reflect the Kauffman skein relations (or obvious properties of tangles).

Definition of BMW algebra

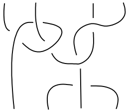
Definition 1

Let R be a ring with parameters ρ, q, δ as before. The BMW algebra $W_{n,R}$ is the R-algebra with generators g_i and e_i for $1 \leq i \leq n-1$, and relations:

1. *The g_i are invertible and satisfy the braid relations.*
2. *(Idempotent relation)* $e_i^2 = \delta e_i$.
3. *(Commutation relations)* $g_i e_j = e_j g_i$ and $e_i e_j = e_j e_i$ if $|i-j| \geq 2$.
4. *(Tangle relations)* $e_i e_{i\pm 1} e_i = e_i$, $g_i g_{i\pm 1} e_i = e_{i\pm 1} e_i$, and $e_i g_{i\pm 1} g_i = e_i e_{i\pm 1}$.
5. *(Crossing relation)* $g_i - g_i^{-1} = (q^{-1} - q)(e_i - 1)$.
6. *(Untwisting relations)* $g_i e_i = e_i g_i = \rho^{-1} e_i$, and $e_i g_{i\pm 1} e_i = \rho e_i$.

BMW algebra – geometric realization

- The BMW algebras can be realized as algebras of tangles (and the inventors of BMW presumably had this in mind from the outset).
- Let R be an integral domain with parameters ρ, q and δ as before. Define the Kauffman tangle algebra $KT_{n,R}$ as the R–algebra of framed (n, n)–tangles in $D \times I$,



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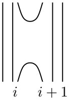
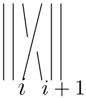
modulo Kauffman skein relations, with multiplication by stacking.

Theorem 2 (Morton-Wasserman, 1989)

The assignments $e_i \mapsto$  *and* $g_i \mapsto$  *determines an isomorphism* $W_{n,R} \xrightarrow{\cong} KT_{n,R}$. *The KT (and hence the BMW) algebra over any integral domain R (with appropriate parameters) is a free R–module of rank*

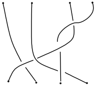
$$(2n-1)(2n-3)\cdots(3)(1).$$

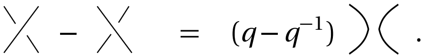
Generic semisimplicity and generic representation theory

The BMW algebras can be thought of as deformations of Brauer algebras. They are generically semisimple, with simple modules of the n-th algebra labelled by Young diagrams of size $n, n-2, n-4, \ldots$. (There is a method due to Wenzl involving realizing the algebras as "repeated Jones basic constructions", starting with the tower of Hecke algebras of the symmetric groups.)

They are also cellular algebras (concept of Graham and Lehrer) and this can be shown using a cellular version of Wenzl's construction (due to Goodman-Graber).

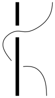
What are affine and cyclotomic BMW algebras?

- What should it mean to affinize the BMW algebras?
- Consider the passage from the ordinary Hecke algebra to the affine Hecke algebra.
- The ordinary Hecke algebra $H_n(q^2)$ is realized geometrically as the the algebra of braids  in the disc cross the interval $(D \times I)$ modulo the Hecke skein relation:

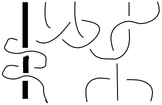


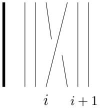
- This is equivalent to the usual presentation of the Hecke algebra by generators and relations.

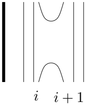
The affine Hecke algebra

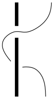
- The affine Hecke algebra $\widehat{H}_n(q^2)$ is the Hecke algebra of the affine symmetric group. It has a presentation (not the Coxeter presentation) with generators $t_1, g_1, \ldots, g_{n-1}$, where
  - $t_1, g_1, \ldots, g_{n-1}$ satisfy the type B braid relations, and
  - $g_1, \ldots, g_{n-1}$ satisfy a quadratic relation.
- The affine Hecke algebra is realized geometrically as the algebra of braids in the annulus cross the interval $(A \times I)$, modulo Hecke skein relations. In this picture, t_1 is a curve wrapping once around the hole in $A \times I$, namely $t_1 =$  .

The affine Kauffman tangle algebra

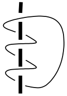
- This suggests a prescription for affinizing the BMW algebra: define the affine Kauffman tangle algebra as the algebra of framed tangles in $A \times I$, modulo Kauffman skein relations.
- We represent framed (n, n)–tangles in $A \times I$, by "affine tangle diagrams":  . The heavy vertical line represents the hole in $A \times I$; we call it the flagpole.
- The affine KT algebra is generated as an algebra by the following affine tangle diagrams:

$G_i =$ 

, $E_i =$ 

, $X_1 =$  .

The affine KT algebra, continued

- There are parameters ρ, q and $\delta = \delta_0$, as before. The subalgebra of $(0,0)$–tangles is a polynomial algebra in the quantities $\boxed{\delta_a = \rho^a}$  for $a \geq 0$ (theorem of Turaev, using invariants of tangles from quantum groups). This polynomial algebra can be absorbed into the ground ring, so the $(0,0)$–tangle algebra is now free of rank 1 over the ground ring.

- Thus the affine KT algebra is defined over a ring with infinitely many parameters, ρ, q and δ_a $(a \geq 0)$. If we put $\boxed{Y_1 = \rho X_1,}$ we then have $\boxed{E_1 Y_1^a E_1 = \delta_a}$ for $a \geq 0$.

The affine BMW algebra

A presentation of the affine KT algebra was proposed by Häring Oldenburg (who introduced the affine and cyclotomic BMW algebras around 1990.)

Definition 3

The affine BMW algebra $\widehat{W}_{n,R}$ over a commutative unital ring R with parameters ρ, q, δ_a $(a \geq 0)$ is the algebra with generators $y_1, e_1, \dots, e_{n-1}, g_1, \dots, g_{n-1}$ and relations:

1. *The e_i's and the g_i's satisfy the BMW relations.*
2. *y_1 is invertible and satisfies $y_1 g_1 y_1 g_1 = g_1 y_1 g_1 y_1$.*
3. *y_1 commutes with g_j and with e_j for $j \geq 2$.*
4. *$e_1 y_1^a e_a = \delta_a e_1$ for $a \geq 0$.*
5. *(Unwrapping relation)* $e_1 y_1 g_1 y_1 = \rho e_1 = y_1 g_1 y_1 e_1$.

Only the last relation is a little mysterious. The geometric version is $\boxed{E_1 X_1 G_1 X_1 = \rho^{-1} E_1,}$ and you can work this out with pictures what this means.

Isomorphism and freeness

An affine Morton–Wasserman type theorem:

Theorem 4 (Goodman-Mosley)

The assignments $e_i \mapsto E_i$, $g_i \mapsto G_i$ and $y_1 \mapsto \rho X_1$ determines an isomorphism of the affine BMW algebra over any suitable ring R onto the affine Kauffman tangle algebra over R. The affine BMW algebra is free over its ground ring of infinite rank.

Remark: For the remainder of the talk, assume $q - q^{-1} \neq 0$ to avoid some complications.

Cyclotomic algebras and representation theory

- Now that we know what the affine BMW algebras are, we can ask about the finite dimensional representation theory. If we work over an algebraically closed field (with appropriate parameters) then in any finite dimensional representation, the "affine generator" y_1 will satisfy a polynomial.
- Define a cyclotomic BMW algebra $W_{n,r,R}(u_1, \ldots, u_r)$ to be the quotient of the affine BMW algebra $\widehat{W}_{n,R}$ by the "cyclotomic relation":

$$(y_1 - u_1) \cdots (y_1 - u_r) = 0.$$

- To study the f.d. representation theory of the affine algebras, we should study all possible cyclotomic quotients.
- Note that the quotient of a cyclotomic BMW algebra by the ideal generated by (any or all of the) e_i's is a cyclotomic Hecke algebra, and a lot is known about the representation theory of affine and cyclotomic Hecke algebras.

Things that should be true

Here are some things that one would expect to be true:

- Any cyclotomic BMW algebra is isomorphic to a cyclotomic version of the KT algebra.
- A cyclotomic BMW algebra $W_{n,r,R}(u_1,\ldots,u_r)$ is free of rank $r^n(2n-1)(2n-3)\cdots(3)(1)$ over R.
- A cyclotomic BMW algebra is generically semisimple, is obtained by repeated Jones basic constructions from the tower of cyclotomic Hecke algebras, and (in the generic semisimple case) has simple modules labelled by r–tuples of Young diagrams of total size n, $n-2$, $n-4$, $\ldots$.
- Cyclotomic BMW algebras are cellular algebras in the sense of Graham and Lehrer.

Admissible parameters

Theorem 5 (Goodman-Mosley, Rui-Xu, Wilcox-Yu)

1. *The statements on the previous slide are all true if and only if the parameters $\rho, q, \Delta = (\delta_a)_{a\geq 0}$ and $u_1, \ldots u_r$ of the cyclotomic BMW algebra satisfy special conditions, called admissibility conditions.*
2. *The admissibility conditions give ρ and the δ_a's as explicit polynomials in the remaining variables, and are equivalent to the 2–strand algebra being free of rank $3r^2$, or to linear independence of $\{e_1, y_1 e_1, \ldots, y_1^{r-1} e_1\}$.*

Theorem 6 (Goodman-Graber)

The cellular version of Wenzl's method with repeated Jones basic constructions also applies here.

The bad news (apparently)

- Some (apparently) bad news: The theorem is true, and not easy, but it isn't exactly what we wanted. We wanted to start with an affine algebra with whatever parameters (ρ, q, Δ) and analyze the representations by looking at all possible cyclotomic quotients.
- More (apparently) bad news: In fact, an affine BMW algebra with arbitrary parameters ρ, q, Δ will have no finite dimensional representations on which $e_1 \neq 0$, i.e. will only have finite dimensional representations factoring through the affine Hecke algebra.
- For "non–Hecke" representations to exist, the parameters must satisfy certain severe conditions. The severe conditions, henceforth known as (SC), are infinitely many polynomial conditions in infinitely many variables, which moreover look to be highly overdetermined.

Resolution of the admissibility problem

Theorem 7 (Goodman)

Consider an affine BMW algebra $\widehat{W}_n$ over an algebraically closed field F, with parameters ρ, q, and $\Delta = (\delta_a)_{a\geq 0}$ (and recall we assume $q - q^{-1} \neq 0$). The following are equivalent:

1. *The severe conditions (SC) hold for the parameters.*
2. *There exist $r > 0$ and $u_1, \ldots, u_r \in F$ such that the parameters ρ, q, Δ, and $u_1, \ldots, u_r$ are admissible.*
3. *$\widehat{W}_n$ admits a finite dimensional module on which e_1 is non–zero.*

Moreover, if there exist $u_1, \ldots, u_r$ such that ρ, q, Δ and $\{u_i\}$ are admissible, then one can, in principle find all other sets $u'_1, \ldots, u'_s$ such ρ, q, Δ and $\{u'_i\}$ are admissible.

Resolution of the admissibility problem, cont.

Theorem 8 (Goodman)

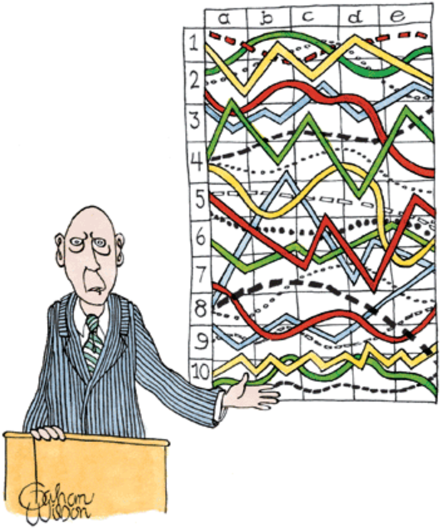
Hypotheses as in theorem 7.

1. *Although the affine BMW algebra may have non–trivial cyclotomic quotients with non–admissible parameters, the structure of these is determined by the admissible case. All the cyclotomic quotients are cellular algebras.*
2. *All simple modules factor either through a cyclotomic Hecke quotient or a cyclotomic BMW quotient with admissible parameters.*

Conclusion

Putting these results together, one can (in principle) find all parameter sets for affine BMW algebras that allow non-trivial cyclotomic quotients and for these one can find all cyclotomic quotients, and all finite dimensional simple modules.

Thank you!



"I'll pause for a moment so you can let this information sink in."